EFRE INTERREG INTEROP PROJECT

Interference Analysis, Modeling, and Emulation

Counting Zeros of Two Tone Signals as Function of Phase and Amplitude

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Contents

Abstract

We derive a closed form formula for the number of zeros of a two tone signal in dependence of the amplitude and phase of the tickle signal.

1 Multi Tone Signals

Multi tone signals considered below are used as input signals in non-linear measurements of microwave devices and circuits. The output spectrum is a superposition of spectral components at integer multiples of the tones at the input of the device. From these measurements, brute force models such as the third order intercept point or more sophisticated models such as X-parameters can be derived.

In the report "Nonlinear Scattering Operator" dated December 23, 2019, the authors follow an alternative approach based on non-linear scattering operators, which are used, e.g., in quantum physics.

In frequency domain the output spectrum can be obtained by applying the non-linear scattering operator to the spectrum of the input multi tone signal. However to calculate the scattering operator in the singular case, i.e. the case where the gain function is defined by a singular distribution [2], we need to know the real zeros of the multi tone signal.

In this case these are possible singularities of the gain function. To find the scattering sequence we must calculate a sequence of proper principal value integrals which depend on these singularities. In addition the multiplicity of these singularities determines if some of this principal value integrals are finite or not, i.e. if the scattering sequence exists at all. This is the main reason, why we are interested in locating the zeros of multi tone signals.

1.1 N+1 tone signals

We denote a signal of the form $s\left(t\right)=\cos\left(t\right)+\sum_{j=1}^{N}\mid\epsilon_j\mid\cos\left(n_jt+\arg\left(\epsilon_j\right)\right)$ as multi tone or more precisely as $N+1$ tone signal. Here $N\in\mathbb{N}$, n_j are mutually different natural numbers unequal to one and ϵ_j are complex, non-zero constants.

Here the remainder $\sum_{j=1}^{N}\mid\epsilon_j\mid\cos\left(n_jt+\arg\left(\epsilon_j\right)\right)$ is called 'tickle signal' though in our setting it is not assumed that $\mid\epsilon_j\mid, j=1,\ldots,N$, are small compared to one. Observe that a multi tone signal $s\left(t\right)$ can be written in the form $s\left(t\right)=\frac{1}{2}e^{-int}p\left(e^{it}\right)$ with a polynomial p of degree $2n$, where $n=\max_{j=1}^{N}n_j$ and

$$p\left(z\right)=z^{n+1}+z^{n-1}+\sum_{j=1}^{N}\epsilon_jz^{n+n_j}+\overline{\epsilon_j}z^{n-n_j}. \tag{1}$$

Hence the real zeros of the 2π periodic multi tone signal $s\left(t\right)$ in the interval $[0,2\pi)$ are in one to one correspondence with the zeros of the polynomial p on the unit circle and are given by the argument defined on $[0,2\pi)$ of this zeros.

It is easily checked that the polynomial p satisfies $z^{2n}\overline{p}\left(\frac{1}{\overline{z}}\right)=p\left(z\right)$, where $\overline{(\cdot)}$ denotes complex conjugation. Thus the polynomial p is selfadjoint [5]. Hence the number of roots of p can be obtained with the help of the Cayley transform q of p

$$q\left(z\right)=\left(z+i\right)^{2n}p\left(\frac{z-i}{z+i}\right)=\left(z+i\right)^{2n}p\left(\mu\left(z\right)\right) \tag{2}$$

which is a real polynomial ([5], Algorithm 5). Here $\mu(z) = \frac{z-i}{z+i}$ is the so called Cayley transform.

Next let us focus on the simplest case of a two tone signal.

1.2 Zeros of two tone signals

In case of a two tone signal $s(t)$ is given by

$$s(t) = \cos(t) + |\,\epsilon\,| \cos(n_1 t + \arg(\epsilon)), \tag{3}$$

$\mathbb{N} \ni n_1 \geq 2$. From Equation (1) we conclude that the associated polynomial p is given by

$$p(z) = \epsilon z^{2n_1} + z^{n_1+1} + z^{n_1-1} + \bar{\epsilon}. \tag{4}$$

We now study the simplest two tone case with $n_1 = 2$. From Equation (4) we get

$$p(z) = \epsilon z^4 + z^3 + z + \bar{\epsilon} \tag{5}$$

and a short calculation reveals that

$$\frac{1}{2}\,q(z) = (1 + \Re(\epsilon))z^4 + 4\Im(\epsilon)\,z^3 - 6\Re(\epsilon)\,z^2 - 4\Im(\epsilon)\,z + \Re(\epsilon) - 1, \tag{6}$$

where $\Re(z)$, $\Im(z)$ denote the real resp. imaginary part of a complex number z. For sake of brevity we write $\Re(\epsilon) = \epsilon_r$ and $\Im(\epsilon) = \epsilon_i$.

1.2.1 The case $\epsilon = -1 + i\epsilon_i$

Then first assume that $\epsilon_r = -1$. Then by Equation (5), $p(1) = 0$, and 1 is one of the zeros of p on the unit circle. By Equation (6) the polynomial $\frac{1}{2}q$ simplifies to the polynomial

$$\frac{1}{2}q(z) = 2\left(2\epsilon_i z^3 + 3z^2 - 2\epsilon_i z - 1\right). \tag{7}$$

From Theorem 1 in [5] we conclude that 1 is a zero of order two of the polynomial p if $\epsilon_i = 0$ and a zero of order one otherwise.

Furthermore in the case of $\epsilon_i = 0$ we see from (7) that the zeros of the polynomial q are $\pm\frac{\sqrt{3}}{3}$, which are mapped by the Cayley transform $\mu(z) = \frac{z-i}{z+i}$ to the zeros $e^{\frac{i2\pi}{3}}$ and $e^{\frac{-i2\pi}{3}} = e^{\frac{i4\pi}{3}}$ of the polynomial p. Summarizing, if $\epsilon = -1$, the real zeros of of the two tone signal $s(t) = \cos(t) - \cos(2t)$ are 0 with order 2, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Next let us assume that $\epsilon = -1 + i\epsilon_i$ with $\mathbb{R} \ni \epsilon_i \neq 0$. Then according to (7) and Theorem 1 in [5] $z = 1$ is a simple zero of the polynomial p and the we need to investigate the real zeros of $\frac{1}{4}q(z) = 2\epsilon_i z^3 + 3z^2 - 2\epsilon_i z - 1$ In the terminology of [3] we calculate $y_N^2 = \frac{1}{4\epsilon_i^4}$ and $h^2 = \frac{1}{4\epsilon_i^4}\left(1 + \frac{4}{3}\epsilon_i^2\right)^3$. Consequently $y_N^2 < h^2$, showing that there are three distinct real roots.

These are obtained in the following way [3]. Set $\theta = \frac{1}{3}\arccos\left(\left(1+\frac{4}{3}\epsilon_i^2\right)^{-\frac{3}{2}}\right)$. Then the three distinct real roots are obtained by

$$\begin{aligned}
z_0 &= -\frac{1}{2\epsilon_i}\left(1+2\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}\cos\left(\theta+\frac{2\pi}{3}\right)\right)\\
z_1 &= -\frac{1}{2\epsilon_i}\left(1+2\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}\cos\left(\theta\right)\right)\\
z_2 &= -\frac{1}{2\epsilon_i}\left(1+2\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}\cos\left(\theta-\frac{2\pi}{3}\right)\right)
\end{aligned}$$

Note that these solutions can be written in hypergeometric form. Indeed setting $\Delta(a) = \left(1+\frac{4}{3}a\right)^{-3}$ for $a \in \mathbb{R}$ and using the definitions of the generalized Chebyshev polynomials of the first and second kind together with the transformation rule

$${}_2F_1\left[\begin{matrix} a\ \ b \\ a+b+\frac{1}{2}\end{matrix};z\right] = {}_2F_1\left[\begin{matrix} 2a\ \ 2b \\ a+b+\frac{1}{2}\end{matrix};\frac{1}{2}\left(1-\sqrt{1-z}\right)\right]$$

for the Gauss hypergeometric function we get

$$\cos(\theta) = {}_2F_1\left[\begin{matrix} \frac{1}{3}\ \ -\frac{1}{3} \\ \frac{1}{2}\end{matrix};\frac{1}{2}\left(1-\sqrt{\Delta(\epsilon_i)}\right)\right] = {}_2F_1\left[\begin{matrix} \frac{1}{6}\ \ -\frac{1}{6} \\ \frac{1}{2}\end{matrix};1-\Delta(\epsilon_i)\right]$$

and

$$\begin{aligned}
\sin(\theta) &= \frac{1}{3}\sqrt{1-\Delta(\epsilon_i)}\;{}_2F_1\left[\begin{matrix} \frac{2}{3}\ \ \frac{4}{3} \\ \frac{3}{2}\end{matrix};\frac{1}{2}\left(1-\sqrt{\Delta(\epsilon_i)}\right)\right]\\
&= \frac{1}{3}\sqrt{1-\Delta(\epsilon_i)}\;{}_2F_1\left[\begin{matrix} \frac{1}{3}\ \ \frac{2}{3} \\ \frac{3}{2}\end{matrix};1-\Delta(\epsilon_i)\right]. \qquad (8)
\end{aligned}$$

Consequently for $a \in \mathbb{R}$ let us define

$$c(a) = {}_2F_1\left[\begin{matrix} \frac{1}{6}\ \ -\frac{1}{6} \\ \frac{1}{2}\end{matrix};1-\Delta(a)\right]$$

and

$$s(a) = \sqrt{\frac{1-\Delta(a)}{3}}\;{}_2F_1\left[\begin{matrix} \frac{1}{3}\ \ \frac{2}{3} \\ \frac{3}{2}\end{matrix};1-\Delta(a)\right].$$

Here we have scaled Expression (8) by $\sqrt{3}$ because we are going to use the addition theorem for trigonometric functions.

Thus we obtain the final hypergeometric form of the real zeros of the polynomial in (7) as

$$z_1 = -\frac{1}{2\epsilon_i}\left(1+2\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}c(\epsilon_i)\right) \qquad (9)$$

and using the addition theorem for trigonometric functions

$$\begin{aligned} z_0 &= -\frac{1}{2\epsilon_i}\left(1-\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}\left(c\left(\epsilon_i\right)+s\left(\epsilon_i\right)\right)\right) \\ z_2 &= -\frac{1}{2\epsilon_i}\left(1-\left(1+\frac{4}{3}\epsilon_i^2\right)^{\frac{1}{2}}\left(c\left(\epsilon_i\right)-s\left(\epsilon_i\right)\right)\right). \end{aligned} \tag{10}$$

Therefore if $\epsilon = -1 + i\epsilon_i$ with $\mathbb{R} \ni \epsilon_i \neq 0$ the two tone signal s in (3) has the four simple real roots 0, $\arg\left(\mu\left(z_i\right)\right) = 2\arctan\left(z_i\right) + \pi$, $i = 0, 1, 2$ for $n_1 = 2$, according to the following Remark.

Remark 1. For real z the Cayley transform simplifies to

$$\mu\left(z\right) = \frac{1}{1+z^2}\left(z^2 - 1 - 2zi\right).$$

Using formula $(A.1)$ on p. 370 in [1] and the known formulas $\arctan\left(-z\right) = -\arctan\left(z\right)$ and $\arctan\left(\frac{1}{z}\right) = sign\left(z\right)\frac{\pi}{2} - \arctan\left(z\right)$, where $sign\left(z\right)$ denotes the sign function we obtain $\arg\left(\mu\left(z\right)\right) = 2\arctan\left(z\right) + \pi$. Here $\arctan\left(\cdot\right)$ is restricted to its principal branch such that $\arg\left(\cdot\right)$ attains its values in the interval $[0, 2\pi)$.

Hence we conclude that counting multiplicity for $\epsilon = -1 + i\epsilon_i$, $\epsilon_i \in \mathbb{R}$, there are always four roots of the polynomial p in (6) on the unit circle and four real roots of s in (3).

1.2.2 Case $\epsilon = \epsilon_r + i\epsilon_i$ with $\epsilon_r \neq -1$

Finally we consider the case $p\left(1\right) \neq 0$, i.e $\epsilon_r \neq -1$, where the number of the roots of the polynomial p in (5) on the unit circle is now identical with the number of real roots of the polynomial q in (6).

First let us assume that $\epsilon_i = 0$, i.e. $\epsilon \in \mathbb{R} \setminus (-1)$. Then it follows from (6) that

$$\frac{1}{4}q\left(z\right) = \left(\epsilon + 1\right)z^4 - 6\epsilon z^2 + \epsilon - 1$$

which is a bi-quadratic polynomial with roots $\pm\sqrt{\frac{3\epsilon\pm\sqrt{1+8\epsilon^2}}{\epsilon+1}}$ and it is easily checked that the real roots are as follows. For $\epsilon = 1$ there are three real roots, where zero is a double root and $\pm\sqrt{3}$ are two single roots. For $|\epsilon| < 1$ we have two single real roots given by $\pm\sqrt{\frac{3\epsilon+\sqrt{1+8\epsilon^2}}{\epsilon+1}}$ and for $|\epsilon| > 1$ there are exactly four real single roots given by $\pm\sqrt{\frac{3\epsilon\pm\sqrt{1+8\epsilon^2}}{\epsilon+1}}$.

The remaining case to consider is where $\epsilon_r \neq -1$ and $\epsilon_i \neq 0$. Here we have to deal with the general quartic polynomial (6).

First we investigate the number of zeros as a function of ϵ. For that purpose we calculate the discriminant of the 4-th dergree polynomial in (6) using the polar form of ϵ and obtain

$$\mathbf{Disc}_z\left(\frac{1}{2}q\right) = -128\, h\left(u,\theta\right),$$

where

$$h\left(u,\theta\right) = 2 + 3\left(1 + 9\cos\left(2\theta\right)\right)u + 96u^2 - 128u^3,$$

with $u = \left|\epsilon\right|^2$ and $\theta = \arg\left(\epsilon\right)$.

Remark 2. Recall the following known facts about the discriminant of 4-th degree polynomial.

The discriminant is zero if and only if two or more roots are equal. If the coefficients are real numbers and the discriminant is negative there are two real roots and two complex conjugate roots, likewise if the discriminant is positive the roots are either all real or all non-real.

Due to the last remark we investigate the zeros of the discriminant or, what amounts to the same, the zeros of the polynomial $h\left(\cdot,\theta\right)$ and are going to show that it has exactly one real zero.

Calculating the discriminant of this 3-rd degree polynomial in u we obtain

$$\mathbf{Disc}_u\left(h\right) = -80621568\left(\cos\left(\theta\right)^2\sin\left(\theta\right)\right)^2 \leq 0.$$

This expression is zero exactly for $\theta \in S = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. However observe that $\epsilon_i \neq 0$ and therefore $\theta = 0$ or $\theta = \pi$ are no possible choices for θ. Next it is easily checked that $u = \frac{1}{4}$ is a triple real root for the remaining two cases where $\epsilon_r = 0$. Here the polynomial (6) reduces to

$$\frac{1}{2}q\left(z\right) = z^4 + 4\epsilon_i z^3 - 4\epsilon_i z - 1 = \left(z^2 - 1\right)\left(z^2 + 4\epsilon_i z + 1\right).$$

Thus we recognize that the polynomial q has the following zeros

$$\left\{-1, 1, -2\epsilon_i - \sqrt{4\epsilon_i^2 - 1}, -2\epsilon_i + \sqrt{4\epsilon_i^2 - 1}\right\} \tag{11}$$

Hence if $\epsilon_i^2 > \frac{1}{4}$ there are four single real roots of the form (11), -1 is a triple and 1 a single root if $\epsilon_i = \frac{1}{2}$, 1 is a triple and -1 a single root if $\epsilon_i = -\frac{1}{2}$, and ± 1 are the only real, single zeros of the polynomial q if $\epsilon_i^2 < \frac{1}{4}$.

Consequently it remains to consider the cases $\theta \in [0, 2\pi) \setminus S$ implying that $\mathbf{Disc}_u\left(h\right) < 0$. However this means that the polynomial $h(\cdot,\theta)$ has exactly one real simple root. This root is determined next.

Using the result in [3] of section 2.2 with $x_N = \frac{1}{4}$, $y_N = h\left(x_N,\theta\right) = \frac{27}{2}\cos^2\left(\theta\right)$, and $h^2 = \left(\frac{27}{2}\right)^2\cos^6\left(\theta\right)$ the simple root is obtained by

$$u_0\left(\theta\right) = \frac{1}{4}\left(1 + \frac{3}{2}\left|\cos\theta\right|^{\frac{2}{3}}\left(\left(1 - \sin\theta\right)^{\frac{1}{3}} + \left(1 + \sin\theta\right)^{\frac{1}{3}}\right)\right). \tag{12}$$

Next observe that $\lim_{u\to\pm\infty} h(u,\theta) = \mp\infty$. Hence $h(u,\theta) \gtrless 0$ for $u \lessgtr u_0(\theta)$ implying $\mathbf{Disc}_z\left(\frac{1}{2}q\right) \lessgtr 0$ for $u \lessgtr u_0(\theta)$. Now Remark 2 shows that there are two real single roots and two complex conjugate roots of the polynomial q if $u < u_0(\theta)$. Moreover since the polynomial q does have at least one real root to be shown immediately, it follows that there are four real simple roots if $u > u_0(\theta)$.

Indeed for the two tone signal (3) we have

$$s(t) = \begin{cases} 1 + |\epsilon| \cos(\arg(\epsilon)), & if \quad t = 0 \\ -|\epsilon| \cos(\arg(\epsilon)), & if \quad t = \frac{\pi}{2} \\ -1 + |\epsilon| \cos(\arg(\epsilon)), & if \quad t = \pi. \end{cases}$$

Therefore if $\arg(\epsilon) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ or $\epsilon = 0$ the two tone signal $s(t)$ has a zero at $t = \frac{\pi}{2}$. Next assume that $\epsilon \neq 0$. Then from the first two equations it follows with the help of the intermediate value theorem that there is a zero in the interval $\left(0, \frac{\pi}{2}\right)$ if $\arg(\epsilon) \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. Using the same argument it follows from the second and third equation that there is a zero in the interval $\left(\frac{\pi}{2}, \pi\right)$ if $\arg(\epsilon) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Since there is a one to one correspondence between real zeros of the two tone signal $s(t)$ and the real roots of the polynomial q we are done.

Finally using Remark 2 again we conclude that the polynomial has a real multiple roots if $u = u_0(\theta)$. To be more precise there can only be a real double root and two real simple roots or a real triple root and one real simple root in this case.

For let us assume there is a real double root z_0 and a possible complex root z_1. By expanding $(z - z_0)^2 (z - z_1)(z - \overline{z_1})$ and comparing the constant term with the constant term in the normalized polynomial in (6) we conclude that $0 \leq z_0^2 |z_1|^2 = \frac{\epsilon_r - 1}{\epsilon_r + 1}$. Consequently $\epsilon_r < -1$ or $\epsilon_r \geq 1$. Since for $|\epsilon| > 1$ we are not on the curve $u_0(\theta)$ we are left with the case $\epsilon = \epsilon_r = 1$. But for this case we have already shown that zero is a double root and $\pm\sqrt{3}$ are two single roots, a contradiction. Therefore our assumption cannot hold and our assertion at the beginning of the paragraph holds true.

Now we can go even one step further and determine for which θ these two kind of multiple roots appear. For this purpose we focus on the determination of the location of the triple roots.

Observe that any triple root of (6) has to be also a double root of the derivative of the polynomial in (6) and consequently the discriminant $\mathbf{Disc}_z\left(\frac{1}{2}q'\right)$, where the prime denotes differentiation, vanishes. By the known formulas for the discriminant of the cubic polynomial and the equations $\epsilon_r = \sqrt{u_0(\theta)} \cos\theta$ resp.

$\epsilon_i = \sqrt{u_0(\theta)} \sin\theta$ we obtain

$$\begin{aligned}
\frac{1}{6912}\mathbf{Disc}_z\left(\frac{1}{2}q'\right) &= 4|\epsilon|^2\left(|\epsilon|^2+\epsilon_r\right)-\epsilon_i^2 \\
&= 4u_0(\theta)(u_0(\theta)+\epsilon_r)-\epsilon_i^2 \\
&= 4u_0(\theta)\left(u_0(\theta)+\sqrt{u_0(t)}\cos\theta-\frac{\sin^2\theta}{4}\right) \\
&= (2u_0(\theta))^2\left(1+\frac{\cos\theta}{\sqrt{u_0(\theta)}}+\left(\frac{\cos\theta}{2\sqrt{u_0(\theta)}}\right)^2-\frac{1}{4u_0}\right) \\
&= (2u_0(\theta))^2\left(\left(1+\frac{\cos\theta}{2\sqrt{u_0(\theta)}}\right)^2-\left(\frac{1}{2\sqrt{u_0}}\right)^2\right) \\
&= (2u_0(\theta))^2\left(1+\frac{1+\cos\theta}{2\sqrt{u_0(\theta)}}\right)\left(1-\frac{1-\cos\theta}{2\sqrt{u_0(\theta)}}\right) \\
&= (u_0(\theta))^2\left(1+\frac{\cos^2\frac{\theta}{2}}{\sqrt{u_0(\theta)}}\right)\left(1-\frac{\sin^2\frac{\theta}{2}}{\sqrt{u_0(\theta)}}\right).
\end{aligned}$$

Thus we are left with the problem of finding all real solutions on the interval $[0, 2\pi)$ of the equation

$$u_0(\theta) = \sin^4\frac{\theta}{2}. \tag{13}$$

By inspection of Equation (13) there are at least three zeros $\theta \in \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. We are going to show that these three are the only one.

First observe that $u_0(\theta) > \frac{1}{4}$ and $\sin^4\frac{\theta}{2} < \frac{1}{4}$ for $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. Hence Equation (13) does not have solutions there. Since $u_0(\theta)$ and $\sin^4\frac{\theta}{2}$ are symmetric functions with respect to π it remains to show that there are no zeros on the interval $\left(\frac{\pi}{2}, \pi\right)$ either. For that purpose we observe that $u_0(\theta) > \sin^4\frac{\theta}{2}$ on $\left(\frac{\pi}{2}, \pi\right)$.

Indeed

$$(1-\sin\theta)^{\frac{1}{3}} + (1+\sin\theta)^{\frac{1}{3}} > 2,$$

implying by Equation (12) that

$$u_0(\theta) > \frac{1}{4}\left(1+3|\cos\theta|^{\frac{2}{3}}\right) \tag{14}$$

for all $\theta \in \left(\frac{\pi}{2}, \pi\right)$. On the other hand

$$\begin{aligned}
\frac{1}{3}\left(4\sin^4\frac{\theta}{2}-1\right) &= \frac{1}{3}\left((1-\cos\theta)^2-1\right) \\
&= |\cos\theta|\frac{|\cos\theta|+2}{3} \\
&< |\cos\theta|
\end{aligned}$$

Consequently

$$\left(\frac{1}{3}\left(4\sin^4\frac{\theta}{2}-1\right)\right)^3 < |\cos\theta|^3 < |\cos\theta|^2$$

on $\left(\frac{\pi}{2},\pi\right)$ leading to the inequality

$$|\cos\theta|^{\frac{2}{3}} > \frac{1}{3}\left(4\sin^4\frac{\theta}{2}-1\right). \tag{15}$$

Finally combining Inequality (14) with Inequality (15) leads to

$$u_0(\theta) > \sin^4\frac{\theta}{2}$$

for all $\theta \in \left(\frac{\pi}{2},\pi\right)$ which was to be shown.

It remains to check that at $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ there appear in fact triple zeros. This is not the case for $\theta = \pi$ since one is the only multiple root in this case which is of order two, cf. the discussion after Equation (7). However in the remaining two cases $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ we have already shown that ∓ 1 are triple roots, cf. the discussion of zeros in (11).

Observe that it follows immediately from Equation (12) that $\max_{\theta\in[0,2\pi)} u_0(\theta) = 1$ and $\min_{\theta\in[0,2\pi)} u_0(\theta) = \frac{1}{4}$ leading to the following theorem.

Theorem 1.1. *Let be $\epsilon \in \mathbb{C}$ with $\epsilon_r \neq -1$ and $\epsilon_i \neq 0$ and let the polynomial q be defined as in Equation* (6).

1) *If $|\epsilon|^2 < \frac{1}{4}$ the polynomial q has two real simple roots.*
2) *If $|\epsilon|^2 > 1$ the polynomial q has four real simple roots.*
3) *If $1/4 \leq |\epsilon|^2 \leq 1$ the number of real roots depends on $arg(\epsilon)$.*
    a) *If $|\epsilon|^2 < u_0(\arg(\epsilon))$ the polynomial q has two real simple roots.*
    b) *In the case of $|\epsilon|^2 = u_0(\arg(\epsilon))$ we have to distinguish two cases.*
        i) *If $\arg(\epsilon) \in \left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$ there are one real triple and on real simple root.*
        ii) *If $\arg(\epsilon) \in [0,2\pi)\setminus\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$ there are one real double root and two real simple roots.*
    c) *If $|\epsilon|^2 > u_0(\arg(\epsilon))$ the polynomial q has four real simple roots.*

1.2.3 Calculation of roots in the remaing cases

As a final step it remains to explicitly calculate the real roots of the polynomial q for $\epsilon_r \neq -1, 0$ and $\epsilon_i \neq 0$. For that purpose a method based an the cubic resolvent of the quartic proposed in [4] seems to be most appropriate because we are interested only in real solutions.

We start with the normalized polynomial $\tilde{q} = \frac{1}{2(1+\epsilon_r)} q$ from Equation (6). According to [4] we need to obtain one real root x_1 of the cubic resolvent

$$R_{\tilde{q}}(\epsilon, z) = z^3 + b(\epsilon) z^2 + c(\epsilon) z + d(\epsilon), \tag{16}$$

where

$$\begin{aligned} b(\epsilon) &= \frac{6\epsilon_r}{1+\epsilon_r}, \\ c(\epsilon) &= -\frac{4\left(4|\epsilon|^2 - 3\epsilon_r^2 - 1\right)}{(1+\epsilon_r^2)^2}, \\ d(\epsilon) &= -\frac{8\epsilon_r\left(4|\epsilon|^2 - \epsilon_r^2 - 3\right)}{(1+\epsilon_r)^3}. \end{aligned}$$

To investigate the real solutions of (16) we use the results in [3] and calculate $x_N = -\frac{2\epsilon_r}{1+\epsilon_r}$, $y_N = R_{\tilde{q}}(\epsilon, x_N) = \frac{16\epsilon_r}{(1+\epsilon_r)^3}$ and $h^2 = \frac{16^2\left(4|\epsilon|^2-1\right)^3}{3^3(1+\epsilon_r)^6}$. Moreover we define

$$\Delta(\epsilon) = \epsilon_r^2 - \left(\frac{4|\epsilon|^2 - 1}{3}\right)^3. \tag{17}$$

In case of $y_N = h^2$ or equivalently $\Delta(\epsilon) = 0$ the cubic resolvent has multiple roots and it is known that this is the case if and only if the same holds true for q. Thus the condition $\Delta(\epsilon) = 0$ is equivalent to $|\epsilon|^2 = u_0(\arg(\epsilon))$. However as can be easily checked by the reader this is nothing else as a change from polar to cartesian coordinates. Now if this condition is satisfied we can set ([3], 2.3)

$$\begin{aligned} x_1(\epsilon) &= x_N + \left(\frac{y_N}{2}\right)^{\frac{1}{3}} \\ &= -\frac{2}{1+\epsilon_r}\left(\epsilon_r + 2\epsilon_r^{\frac{1}{3}}\right). \end{aligned} \tag{18}$$

which is the real simple root of the cubic resolvent. Here and in the following for $a \in \mathbb{R}$ the expression $a^{\frac{1}{3}}$ denotes the real cubic root $sign(a)|a|^{\frac{1}{3}}$. The reason for choosing the simple root is due to the fact that it is a special case of the following one.

In the case $y_N^2 > h^2$, i.e. $\Delta(\epsilon) > 0$ or equivalently $|\epsilon|^2 < u_0(\arg(\epsilon))$ the cubic resolvent has exactly one real root. Hence we set ([3], 2.2)

$$x_1(\epsilon) = -\frac{2}{1+\epsilon_r}\left(\epsilon_r + \left(\epsilon_r + \sqrt{\Delta(\epsilon)}\right)^{\frac{1}{3}} + \left(\epsilon_r - \sqrt{\Delta(\epsilon)}\right)^{\frac{1}{3}}\right) \tag{19}$$

In the final case $y_N^2 < h^2$, i.e. $\Delta(\epsilon) < 0$ or equivalently $|\epsilon|^2 > u_0(\arg(\epsilon))$ the cubic resolvent has exactly three real roots. Hence we set ([3], 2.4)

$$\begin{aligned} x_1(\epsilon) &= -\frac{2}{1+\epsilon_r}\left(\epsilon_r + 2\sqrt{\frac{4|\epsilon|^2-1}{3}}\cos\left(\frac{1}{3}\arccos\frac{\epsilon_r}{\left(\frac{4|\epsilon|^2-1}{3}\right)^{\frac{3}{2}}}\right)\right) \\ &= -\frac{2}{1+\epsilon_r}\left(\epsilon_r + 2\sqrt{\frac{4|\epsilon|^2-1}{3}}\,{}_2F_1\left[\begin{matrix}\frac{1}{6} & -\frac{1}{6} \\ & \frac{1}{2}\end{matrix};-\frac{\Delta(\epsilon)}{\left(\frac{4|\epsilon|^2-1}{3}\right)^3}\right]\right) \end{aligned} \quad (20)$$

where we used similar arguments as for $\epsilon = -1 + i\epsilon_i$ to obtain the hypergeometric form of the root. In the following depending in the sign of $\Delta(\epsilon)$ the root $x_1(\epsilon)$ has to be choosen according to (18), (19) or (20).

Next to calculate the roots of (6) only Case I in [4] is relevant. With

$$\begin{aligned} A(\epsilon) &= \frac{4\epsilon_i}{1+\epsilon_r}, \\ B(\epsilon) &= \frac{-6\epsilon_r}{1+\epsilon_r}, \\ C(\epsilon) &= -A(\epsilon), \\ D(\epsilon) &= \frac{\epsilon_r - 1}{1+\epsilon_r} \end{aligned}$$

find

$$m(\epsilon) = \sqrt{\frac{1}{4}A^2(\epsilon) - B(\epsilon) + x_1(\epsilon)}$$

and

$$n(\epsilon) = \begin{cases} \sqrt{\frac{1}{4}x_1^2(\epsilon) - D(\epsilon)}, & if \quad m(\epsilon) = 0 \\ \frac{A(\epsilon)x_1(\epsilon) - 2C(\epsilon)}{4m(\epsilon)} = \frac{A(\epsilon)}{4m(\epsilon)}(x_1(\epsilon)+2), & if \quad m(\epsilon) > 0. \end{cases}$$

Observe that $m(\epsilon)$ cannot be imaginary because there are no solutions in the form of Case II in [4]. Now define

$$\begin{aligned} \alpha(\epsilon) &= \frac{1}{2}A^2(\epsilon) - x_1(\epsilon) - B(\epsilon) \\ \beta(\epsilon) &= 4n(\epsilon) - A(\epsilon)m(\epsilon) \\ \gamma(\epsilon) &= \sqrt{\alpha(\epsilon) + \beta(\epsilon)} \\ \delta(\epsilon) &= \sqrt{\alpha(\epsilon) - \beta(\epsilon)}. \end{aligned}$$

Then if $\Delta(\epsilon) < 0$ resp. $|\epsilon|^2 > u_0(\arg(\epsilon))$ the four real simple roots of (6) are given by

$$
\begin{aligned}
X_1(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)+m(\epsilon)+\gamma(\epsilon)}{2}, \\
X_2(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)-m(\epsilon)+\delta(\epsilon)}{2}, \\
X_3(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)+m(\epsilon)-\gamma(\epsilon)}{2}, \\
X_4(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)-m(\epsilon)-\delta(\epsilon)}{2}.
\end{aligned} \tag{21}
$$

If $\Delta(\epsilon) > 0$ resp. $|\epsilon|^2 < u_0(\arg(\epsilon))$ either $\gamma(\epsilon)$ or $\delta(\epsilon)$ is imaginary. Therefore we define

$$
\nu(\epsilon) = \sqrt{\max(\alpha(\epsilon)+\beta(\epsilon), \alpha(\epsilon)-\beta(\epsilon))} = \sqrt{\alpha(\epsilon)+|\beta(\epsilon)|}
$$

and the step function

$$
\theta(x_1, x_2) = \begin{cases} 1, & if \quad x_1 > x_2 \\ -1, & if \quad x_1 \le x_2. \end{cases}
$$

to obtain the two real simple roots in the form

$$
\begin{aligned}
\tilde{X}_1(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)-\theta(\alpha,\beta)\,m(\epsilon)+\nu(\epsilon)}{2}, \\
\tilde{X}_2(\epsilon) &= \frac{-\frac{1}{2}A(\epsilon)-\theta(\alpha,\beta)\,m(\epsilon)-\nu(\epsilon)}{2}.
\end{aligned} \tag{22}
$$

If $\Delta(\epsilon) = 0$ resp. $|\epsilon|^2 < u_0(\arg(\epsilon))$ the two real, simple roots are obtained by $\tilde{X}_1$ and $\tilde{X}_2$ in (22) and the real double root (recall that we assume $\epsilon_r \neq 0$) is obtained by

$$
\tilde{X}_3(\epsilon) = \frac{-\frac{1}{2}A(\epsilon)+\theta(\alpha,\beta)\,m(\epsilon)}{2}. \tag{23}
$$

Again the Cayley transform μ gives the zeros of the polynomial p on the unit circle and finally by taking arguments and taking into account Remark 1 we can calculate the zeros of the two tone function $s(t)$ in (3) with $n_1 = 2$ and $\epsilon_r \neq -1$. Moreover it is easily checked that all particular cases considered are also covered by the general case considered in this section. Therefore our results can be summarized in the following figures.

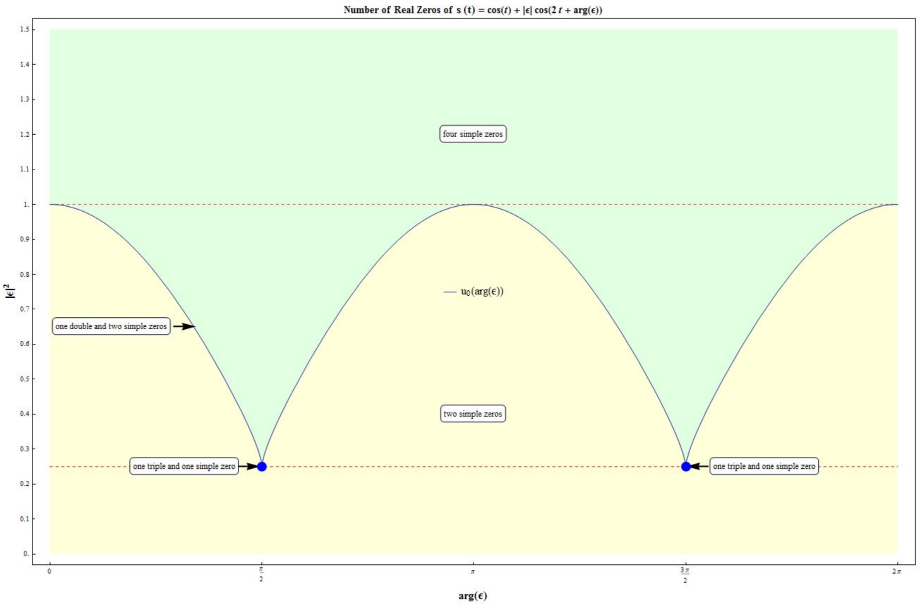


Figure 1: Zero separating curve $u_0\left(\arg\left(\epsilon\right)\right)$ of Equation (12)

In the following decision tree we summarize the different cases considered for calculating the real zeros of the two tone signal $s\left(t\right)$ in (3) where $n_1 = 2$ together with the zeros obtained. For brevity we set $\nu\left(z\right) = 2\arctan\left(z\right) + \pi$ and write $\left(t\right)_n$ for a zero of multiplicity n and for simple zeros we write simply t. Recall that ν is the argument function of the Cayley transform for real z.

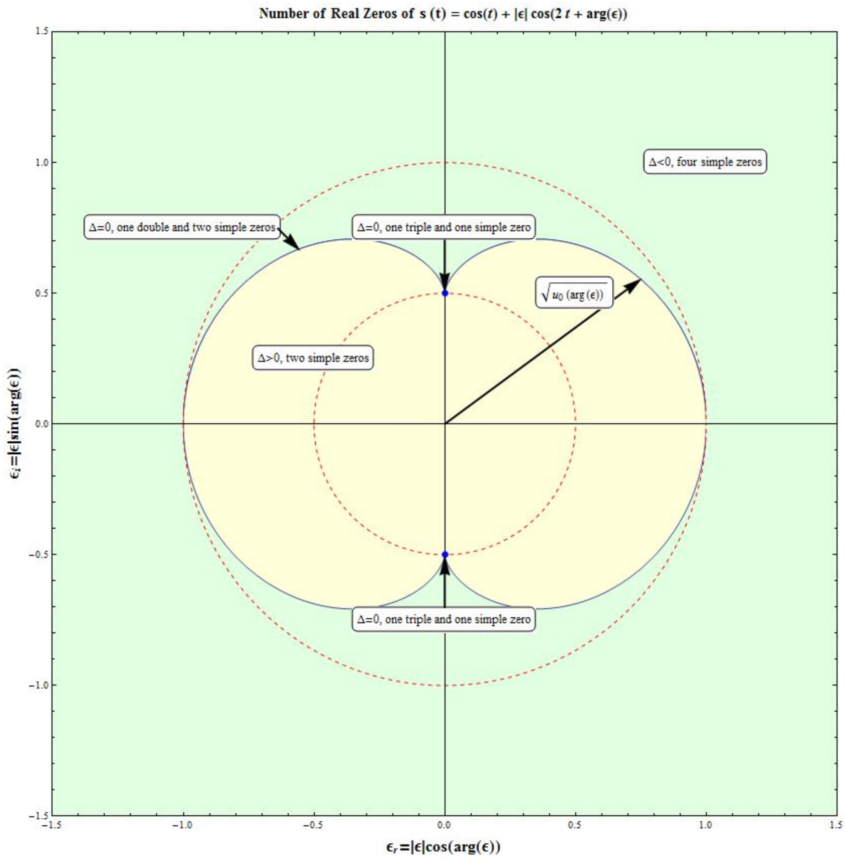
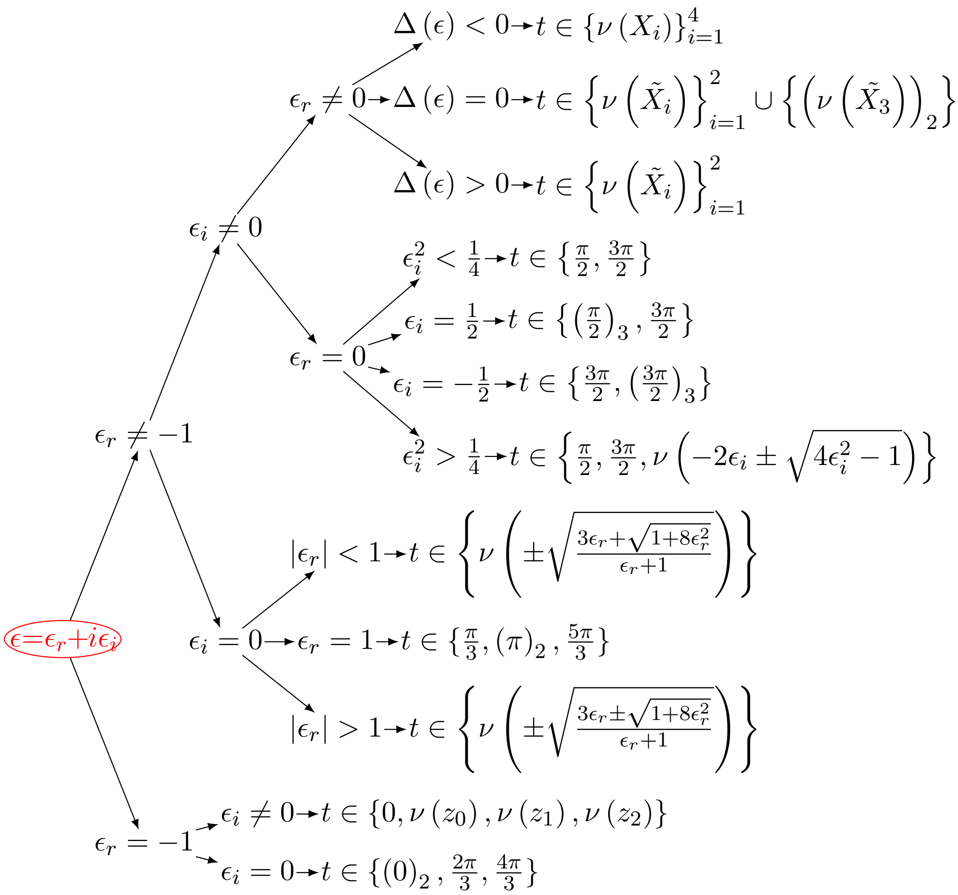


Figure 2: Zero separating curve $u_0\left(\arg\left(\epsilon\right)\right)$ in cartesian coordinates



Here $\Delta(\epsilon)$ is given by (17), $X_i(\epsilon)$, $i = 1, 2, 3, 4$, have to be taken from (21), $\tilde{X}_i(\epsilon)$, $i = 1, 2$, from (22), $\tilde{X}_3(\epsilon)$ from (23), and z_i, $i = 0, 1, 2$ from (9) and (10).

1.3 Conclusion

Consider the periodic signal $s(t) = a\cos(\omega t + \phi_1) + b\cos(2\omega t + \phi_2)$, with $\omega \in \mathbb{R}$ and $a, b > 0$ where the second summand is thought as a perturbation of the first one. By the transformation $\tau = \omega t + \phi_1$ the signal $s(t)$ is transformed into a scaled two tone signal $s(\tau) = a(\cos(\tau) + \epsilon\cos(2\tau + \Delta\phi))$ where $\epsilon = \frac{b}{a}$ and $\Delta\phi = \phi_2 - 2\phi_1$.

Usually the phase ϕ_2 of the perturbation is to be considered random or unknown and one tries to hold the perturbation amplitude b in a certain range. For sure this amplitude would be considered to large if the number of zeros of the sum signal $s(t)$ is greater than two. We have shown that independent of the phase ϕ_2 this will not happen if $b[dB] < a[dB] - 20\log_{10} 2$ and this is optimal.

Moreover in this particular case the two real simple zeros $t_i \in \left[0, \frac{2\pi}{\omega}\right)$, $i = 1, 2$, of the periodic signal $s(t)$ are given by

$$t_i = \frac{\tau_i - \phi_1}{\omega} \mod \frac{2\pi}{\omega},$$

where $\tau_i \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ if $\Delta\phi \bmod(2\pi) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $\tau_i = 2\arctan\left(\tilde{X}_i\right) + \pi$ otherwise, where $\tilde{X}_i$, $i = 1, 2$ are obtained from (22).

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