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Nonlinear Scattering Operator

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Abstract

We present an abstract definition of a nonlinear scattering operator in L^2 spaces which is based on the assumption that there is some nonlinear operator in time domain which describes the input output relationship of the system under consideration. This scattering operator coincides in a certain sense made precise in this work with the classical scattering matrix resp. S parameters in the absence of nonlinearities. It is determined by convolution with an in general not square summable sequence which depends nonlinearly on the input. Several examples are given where this scattering sequence is calculated. The notion of the scattering operator is extended to an 2 port system where the extension to an general n - port system follows similar lines.

The main advantage of this operator is that it is does not depend on any linearization process, is not defined via Volterra series techniques and can exist even globally. This is demonstrated by an application to the nonlinear Schrödinger equation. In the case of a Kerr nonlinearity we get the behavioural equation in frequency domain in closed form and solve the problem of constructing a nonlinear scattering matrix for this given behavioural equation. This matrix is finally regularized such that it depends continously on the input data. Linearization of the regularized nonlinear scattering matrix leads to the X parameters for the nonlinear Schrödinger equation.

1 Nonlinear Scattering

1.1 The Scattering Operator associated to a Nonlinear Operator on L^2

In the discussion of this section we focus on $L^2(-\pi,\pi)$ the space of possible complex valued square integrable functions on $[-\pi,\pi]$. For any other interval than $[-\pi,\pi]$ we can obtain the same conclusions by a change of variable.

Let

$$\mathcal{F}: \begin{array}{c} dom(\mathcal{F}) \subset L^2(-\pi,\pi) \to L^2(-\pi,\pi) \\ \psi \mapsto \mathcal{F}(\psi) \end{array}$$

be a nonlinear operator on $L^2(-\pi,\pi)$ with domain $dom(\mathcal{F})$ and denote by $\hat{\psi}$ the $l^2(\mathbb{Z})$ sequence of Fourier coefficients $\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(t)e^{-int}\,dt$ of the L^2 function ψ.

Now how are the input spectrum $a = \hat{\psi}$ and output spectrum $b = \widehat{\mathcal{F}(\psi)}$ related? In case the operator $\mathcal{F}$ is a bounded linear operator this can be answered easily and is well known.

In fact using the Fourier series representation of ψ, continuity, and linearity of $\mathcal{F}$ we obtain $b = Sa$ where the linear scattering matrix S is given by $S = \left(\widehat{\mathcal{F}(e^{int})}(k)\right)_{k,n\in\mathbb{Z}}$. For example the linear and bounded shift operator by t_0 diagonalizes in the frequency domain and S is given by $S = diag_{n\in\mathbb{Z}}\left(e^{int_0}\right)$.

To proceed in the nonlinear case the idea is to factorize the function $\mathcal{F}(\psi)$ in the form $\frac{\mathcal{F}(\psi)}{\psi}\psi = M_{\mathcal{F}}(\psi)\psi$ and to find an operator representation for the multiplication with the function $M_{\mathcal{F}}(\psi)$ in frequency domain.

To be more precise suppose $m_{\mathcal{F}}(\psi) = \widehat{M_{\mathcal{F}}(\psi)}$ exists in the sense of distributions. Then on the domain

$$\mathcal{D}_{\mathcal{F}} = dom(S_{\mathcal{F}}(\psi)) = \left\{v \in l^2(\mathbb{Z}) \mid m_{\mathcal{F}}(\psi) * v \in l^2(\mathbb{Z})\right\}$$

we define the linear operator $S_{\mathcal{F}}(\psi) = m_{\mathcal{F}}(\psi)*$, where $*$ denotes discrete convolution, i.e. $(m_{\mathcal{F}}(\psi) * v)(n) = \sum_{l\in\mathbb{Z}} m_{\mathcal{F}}(\psi)(l)\, v(n-l)$ for all $n \in \mathbb{Z}$. Observe that $\mathcal{D}_{\mathcal{F}}$ is non empty since always $a \in \mathcal{D}_{\mathcal{F}}$. Moreover by construction we have $b = S_{\mathcal{F}}(\psi)\,a$ where a and b are defined as above.

Next denote by $\mathcal{L}(l^2)$ the set of not necessarily everywhere defined linear operators on $l^2(\mathbb{Z})$ and for $x \in l^2(\mathbb{Z})$ define $\mathcal{L}_x(l^2) = \{T \in \mathcal{L}(l^2) \mid x \in dom(T)\}$. Two operators $T_1, T_2 \in \mathcal{L}_x(l^2)$ are called equivalent, $T_1 \sim T_2 \Leftrightarrow T_1 x = T_2 x \Leftrightarrow \exists N \in \mathcal{L}_x(l^2) \quad with \quad T_2 = T_1 + N \quad and \quad x \in Ker(N)$ and it is easily checked that this defines indeed an equivalence relation on $\mathcal{L}_x(l^2)$. Let us denote the set of all equivalence classes $\mathcal{L}_x(l^2)/\sim$ by π_x and equivalence classes by $[\cdot]_\sim$.

Finally we define the scattering operator as follows. Set

$$dom(S_{\mathcal{F}}) = \{\psi \in dom(\mathcal{F}) \mid m_{\mathcal{F}}(\psi) \text{ exists in the sense of distributions}\}.$$

Then the scattering operator $S_{\mathcal{F}}$ is defined by

$$S_{\mathcal{F}}: \begin{array}{l} dom(S_{\mathcal{F}}) \to \cup_{\psi\in dom(S_{\mathcal{F}})}\pi_{\hat{\psi}} \\ \psi \mapsto [S_{\mathcal{F}}(\psi)]_\sim. \end{array}$$

To avoid an overboarding notation we omit an explicit notation of equivalence classes but should have in mind that the symbol $S_{\mathcal{F}}(\psi)$ actually represents an equivalence class of linear operators.

Remark 1. Observe that the matrix representation of $S_{\mathcal{F}}(\psi)$ with respect to the canononical basis in $l^2(\mathbb{Z})$ is given by the double infinite Laurent matrix

$$S_{\mathcal{F}}(\psi)=\left(\begin{array}{ccc|cccc} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & m_0 & m_{-1} & m_{-2} & m_{-3} & m_{-4} & \ddots \\ \ddots & m_1 & m_0 & m_{-1} & m_{-2} & m_{-3} & \ddots \\ \hline \ddots & m_2 & m_1 & m_0 & m_{-1} & m_{-2} & \ddots \\ \ddots & m_3 & m_2 & m_1 & m_0 & m_{-1} & \ddots \\ \ddots & m_4 & m_3 & m_2 & m_1 & m_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right),$$

where we have set $m = m_{\mathcal{F}}(\psi)$ for short. $m_{\mathcal{F}}(\psi)$ is called the $\mathcal{F}$ adapted scattering sequence or only the scattering sequence.

Example 1.1. *Suppose that the basis functions* e^{int}, $n \in \mathbb{Z}$, *are elements of* $dom(\mathcal{F})$. *Then by definition* $m_{\mathcal{F}}(e^{int})$ *is an* $l^2(\mathbb{Z})$ *sequence and therefore the basis functions are in the domain of the scattering operator* $S_{\mathcal{F}}$ *and moreover* $S_{\mathcal{F}}(\psi)$ *defines a bounded linear operator from* $l^2(\mathbb{Z})$ *into* $l^\infty(\mathbb{Z})$ *with*

$$\|S_{\mathcal{F}}(e^{int})\|_{B(l^2,l^\infty)} \leq \|\mathcal{F}(e^{int})\|_{L^2}$$

by the Cauchy Schwarz inequality.

Next suppose that $\mathcal{F}$ *is bounded and linear s.t. the linear scattering matrix exists. Then by construction* $S \sim S_{\mathcal{F}}(\psi)$ *for all* $\psi \in dom(S_{\mathcal{F}})$. *So actually we are dealing only with the equivalence class* $[S]_\sim$ *and* $S_{\mathcal{F}}$ *is actually a constant map. Moreover though* $S_{\mathcal{F}}$ *is not defined on the whole* L^2 *nethertheless it determines* S *uniquely since as is easily checked the columns of the* S *matrix are obtained by* $S_{\mathcal{F}}(e^{int})\,\delta_n$ *where* δ_n *denotes the* n*-th canonical basis vector*

$$\delta_n(k)=\begin{cases} 1 & if \quad k=n \\ 0 & if \quad k\neq n. \end{cases}$$

Consequently the operator $S_{\mathcal{F}}$ *extends the notion of the scattering matrix to the nonlinear case including infinite dimensions.*

Example 1.2. *Suppose* $\psi \in dom(\mathcal{F})$ *satisfies* $\mathcal{F}(\psi) = \lambda\psi$ *for some* $\lambda \in \mathbb{C}$. *Then* $\psi \in dom(S_{\mathcal{F}})$ *and by definition* $S_{\mathcal{F}}(\psi) = \lambda\delta_0* = \lambda\mathbf{id}$, *where* $\mathbf{id}$ *is the identity map in* $l^2(\mathbb{Z})$. *For the bounded linear shift operator* $\mathcal{F}(\psi)(t) = \psi(t+t_0)$ *the basis function* e^{int}, $n \in \mathbb{Z}$, *is an eigenfunction with eigenvalue* e^{int_0}. *Therefore by Example 1.1 the columns of the linear scattering matrix are given by* $e^{int_0}\delta_n$ *resp.* $S = diag_{n\in\mathbb{Z}}(e^{int_0})$ *as mentioned at the beginning of this section.*

The following example demonstrates why we have to deal with equivalence classes.

Example 1.3. *Again consider the shift operator but now applied to the function* $\psi(t) = \cos t$*. From Example 1.2 we know that* $S_{\mathcal{F}}(\cos(\cdot)) \sim S = diag_{n\in\mathbb{Z}}\left(e^{int_0}\right)$*. This is now checked directly. By the addition theorem for the cosine we have* $S_{\mathcal{F}}(\cos(\cdot)) = \left(\cos t_0 \delta_0 - \sin(t_0)\, \widehat{\tan t}\right) *$ *with*

$$\widehat{\tan t}(n) = \begin{cases} -i(-1)^{\frac{n}{2}} & \text{if } n < 0 \text{ is even} \\ i(-1)^{\frac{n}{2}} & \text{if } n > 0 \text{ is even} \\ 0 & \text{if } n = 0 \text{ or } n \text{ is odd.} \end{cases}$$

Applying the input spectrum $a = \widehat{\cos t} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ *we obtain* $S_{\mathcal{F}}(\cos t)\, a = Sa$*, i.e.* $S_{\mathcal{F}}(\cos t) \sim S$*.*

Example 1.4. *Next we investigate the non linear operator* $\mathcal{F}(\psi)(t) = \sqrt{1 - |\psi(t)|^2}$ *with* $dom(\mathcal{F}) = \{\psi \in L^2(-\pi,\pi) \mid |\psi(t)| \le 1]\}$*. Since* $\mathcal{F}\left(e^{int}\right) = 0$*,* $n \in \mathbb{Z}$*, we note that* $e^{int} \in dom(S_{\mathcal{F}})$ *with* $S_{\mathcal{F}}\left(e^{int}\right) = 0$*.*

Moreover $\cos t \in dom(\mathcal{F})$ *with* $\mathcal{F}(\cos t) = |\sin t|$ *and in addition* $\cos t \in dom(S_{\mathcal{F}})$ *with* $S_{\mathcal{F}}(\cos t) = m_{\mathcal{F}}(\cos t) *$*, where*

$$m_{\mathcal{F}}(\cos t)(n) = \begin{cases} \frac{1}{2\pi} s_{\frac{|n|-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

with $s_n = (-1)^n 4\left(1 + 2\sum_{k=1}^{n} \frac{(-1)^k}{1-4k^2}\right)$*. Thus* $S_{\mathcal{F}}$ *is a non constant map which can only happen because* $\mathcal{F}$ *is nonlinear.*

Example 1.5. *Let* $\mathcal{F}$ *be defined as in the last example. There are functions with non zero mean value which belong to* $dom(S_{\mathcal{F}})$*, e.g take the constant function* $\psi \equiv 1$ *in the last example.*

On the other side it can happen due to a non zero mean value of $\psi \in L^2(-\pi,\pi)$ *that* ψ *is not an element of* $dom(S_{\mathcal{F}})$*. However we can remove this mean value* $c = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t)\, dt$ *from* ψ *and obtain a slight modification to the above discussion.*

To be more precise we have to indroduce the c *modified scattering sequence* $m_{\mathcal{F}}^c(\psi) = \widehat{\left(\frac{\mathcal{F}(\psi)}{\psi - c}\right)}$ *and the scattering operator* $S_{\mathcal{F}}^c$ *with* $S_{\mathcal{F}}^c(\psi) = m_{\mathcal{F}}^c(\psi) *$*. The relation between input spectrum* a *and output spectrum* b *is then given by*

$$b = S_{\mathcal{F}}^c(\psi)\, a - c \cdot m_{\mathcal{F}}^c(\psi).$$

For example this is the case for $\psi(t) = \frac{1+\cos t}{2} = \cos^2\left(\frac{t}{2}\right)$*, where* $\psi \in dom(\mathcal{F})$ *with* $\mathcal{F}(\psi)(t) = \sqrt{1 - \cos^4\left(\frac{t}{2}\right)}$*,* $c = \frac{1}{2}$*, and* $\psi \in dom(S_{\mathcal{F}}^c)$ *with* $S_{\mathcal{F}}^c(\psi) =$

$m^c_{\mathcal{F}}(\psi) *$, *where*

$$m^c_{\mathcal{F}}(\psi)(n) = \frac{2}{\pi}\begin{cases} c^e_{\frac{|n|}{2}}\sqrt{2} + d^e_{\frac{|n|}{2}}\ln\left(1+\sqrt{2}\right) + c_0 & \textit{if } n \textit{ is even} \\ c^o_{\frac{|n|-1}{2}}\sqrt{2} + d^o_{\frac{|n|-1}{2}}\ln\left(1+\sqrt{2}\right) & \textit{if } n \textit{ is odd,} \end{cases}$$

with firstly $c_0 = (-1)^{\frac{n}{2}}\frac{\sqrt{3}}{2}\left(\ln\left(5-2\sqrt{6}\right)\right)$ *and*

$$\begin{aligned} c^e_n &= -2(-1)^n\sum_{k=0}^{n-1}(-1)^k c_{2k+1}, \\ c^o_n &= (-1)^n\left(1+2\sum_{k=1}^{n}(-1)^k c_{2k}\right) \textit{ with} \\ c_n &= (-1)^n\left(1-2n\sum_{k=1}^{n}(-4)^{k-1}\frac{1}{k}\binom{n+k-1}{2k-1}a_k\right), \textit{ where} \qquad (1) \\ a_n &= \frac{1}{n+1} + (-1)^n\frac{\left(\frac{1}{2}\right)_n}{2(n+1)!}\sum_{k=1}^{n}(-1)^k\frac{(k-1)!}{\left(\frac{1}{2}\right)_k}. \end{aligned}$$

Secondly

$$\begin{aligned} d^e_n &= (-1)^n\left(1-2\sum_{k=0}^{n-1}(-1)^k d_{2k+1}\right), \\ d^o_n &= (-1)^n\left(1+2\sum_{k=1}^{n}(-1)^k d_{2k}\right) \textit{ with} \\ d_n &= (-1)^n\left(1-2n\sum_{k=1}^{n}(-4)^{k-1}\frac{1}{k}\binom{n+k-1}{2k-1}b_k\right), \textit{ where} \qquad (2) \\ b_n &= (-1)^n\frac{(2n-1)!!}{2^n(n+1)!}. \end{aligned}$$

Here $(\cdot)_n$ *denotes the Pochhammer symbol. Observe that the output spectrum is obtained by*

$$b = \frac{1}{\pi}\left(c_{|n|}\sqrt{2} + d_{|n|}\ln\left(1+\sqrt{2}\right)\right)_{n\in\mathbb{Z}},$$

where c_n *and* d_n *are defined as in Equation 1 resp. 2.*

However the main point is that b *can be obtained by linearly mapping the input spectrum* a *to the output spectrum* b *by the scattering operator* $S_{\mathcal{F}}(\psi)$ *as in the linear case.*

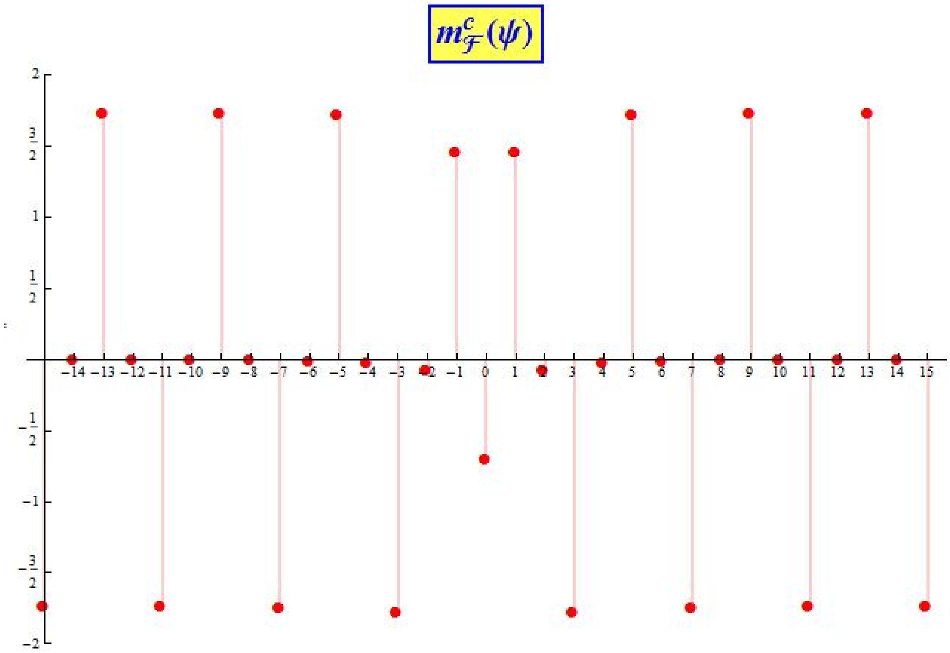

Figure 1: Scattering sequence of Example 1.5

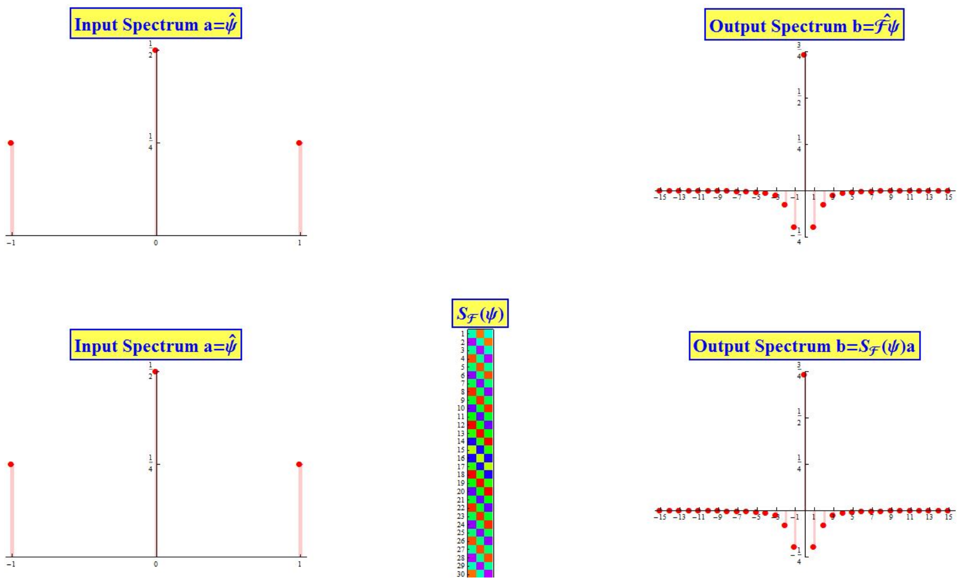

Figure 2: Comparism of output spectra of Example 1.5

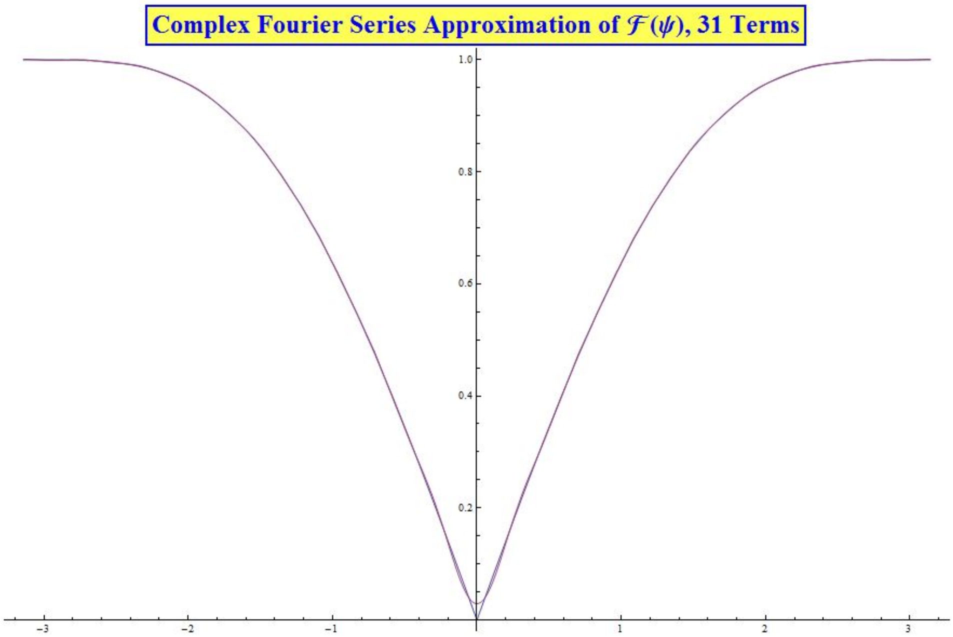

Figure 3: $\mathcal{F}(\psi)$ of Example 1.5 and its Fourier series approximation

1.2 Two Port Nonlinear Scattering Operator

Our next task is the extension of the scattering operator introduced in the last Subsection 1.1 to multiple ports. For sake of simplicity this is done only for the case of two ports. However the extension to more than two ports should be obvious. The state space consists now of two copies of $L^2(-\pi,\pi)$, i.e. $\psi \in L^2(-\pi,\pi)^2 = L^2(-\pi,\pi) \oplus L^2(-\pi,\pi)$.

Next denote by $\mathbb{M}_2(X)$ or simply $\mathbb{M}_2$, if the set X is clear from the context, the set of 2×2 matrices with entries in X. The nonlinear operator $\mathcal{F}$ is now replaced by the matrix valued nonlinear operator

$$\mathcal{F}: \begin{array}{c} dom(\mathcal{F}) \subset L^2(-\pi,\pi)^2 \to \mathbb{M}_2\left(L^2(-\pi,\pi)\right) \\ \psi \mapsto \begin{pmatrix} \mathcal{F}_{11}(\psi) & \mathcal{F}_{12}(\psi) \\ \mathcal{F}_{21}(\psi) & \mathcal{F}_{22}(\psi) \end{pmatrix} \end{array}.$$

Here

$$\mathcal{F}_{ik}: \begin{array}{c} dom(\mathcal{F}_{ik}) \subset L^2(-\pi,\pi)^2 \to L^2(-\pi,\pi) \\ \psi \mapsto \mathcal{F}_{ik}(\psi) \end{array}$$

$i,k = 1,2$, are itself nonlinear operators with

$$dom(\mathcal{F}_{ik}) = \{\psi \in L^2(-\pi,\pi)^2 \mid \mathcal{F}_{ik}(\psi) \in L^2(-\pi,\pi)\}$$

and $dom(\mathcal{F}) = \cap_{i,k=1}^2 dom(\mathcal{F}_{ik})$. The interpretation of $\mathcal{F}_{ik}$ is as follows.

$\mathcal{F}_{11}$ describes the reponse at port 1 due to the presence of the input signal ψ_1 at port one , the first component of ψ. However since this response could be different for a different input signal ψ_2 at port 2, the second component of ψ, $\mathcal{F}_{11}$ must be considered as a function of ψ. In this sense ψ_2 is considered as a parameter function for $\mathcal{F}_{11}$. Similarly $\mathcal{F}_{12}$ describes the (cross) response at port 1 due to the presence of the input signal ψ_2 at port two. Again this response could be different for a different input signal ψ_1 at port 1. Thus $\mathcal{F}_{12}$ must be considered as a function of ψ, too. In this sense ψ_1 is considered as a parameter function of $\mathcal{F}_{12}$. Similar considerations hold for the nonlinear operators $\mathcal{F}_{21}$ and $\mathcal{F}_{22}$, where the roles of ψ_1, ψ_2 have to be interchanged.

Now the output of the two port is given by some $\phi \in L^2(-\pi,\pi)^2$. On the other side having two response functions on each port at our disposal how they are related to the ouput function ϕ?

For this purpose we assume that the responses superimpose additively such that the output at port one is given as $\phi_1 = \mathcal{F}_{11}(\psi) + \mathcal{F}_{12}(\psi)$ and similarly at port 2 as $\phi_2 = \mathcal{F}_{21}(\psi) + \mathcal{F}_{22}(\psi)$. Hence the output spectrum $b = \widehat{\phi} \in l^2(\mathbb{Z})^2$ of the two ports is given by

$$b = \begin{pmatrix} \widehat{\phi_1} \\ \widehat{\phi_2} \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{F}_{11}(\psi)} + \widehat{\mathcal{F}_{12}(\psi)} \\ \widehat{\mathcal{F}_{21}(\psi)} + \widehat{\mathcal{F}_{22}(\psi)} \end{pmatrix}. \tag{3}$$

To proceed we introduce the scattering operator from the last Subsection 1.1 in its most general form as defined in Example 1.5.

For that purpose we assume that $\psi_1 \in dom\left(S^{c_1}_{\mathcal{F}_{11}(\cdot,\psi_2)}\right) \cap dom\left(S^{c_1}_{\mathcal{F}_{21}(\cdot,\psi_2)}\right)$ and $\psi_2 \in dom\left(S^{c_2}_{\mathcal{F}_{12}(\psi_1,\cdot)}\right) \cap dom\left(S^{c_2}_{\mathcal{F}_{22}(\psi_1,\cdot)}\right)$ and write $m_{ij} = m^{c_j}_{\mathcal{F}_{ij}}(\psi_j)$, $i,j = 1,2$, for the modified scattering sequences. Thus by the definition of the corresponding one port scattering operators we obtain frem Equation 3

$$\begin{aligned} b &= \begin{pmatrix} S^{c_1}_{\mathcal{F}_{11}(\cdot,\psi_2)}\widehat{\psi_1} - c_1 m_{11} + S^{c_2}_{\mathcal{F}_{12}(\psi_1,\cdot)}\widehat{\psi_2} - c_2 m_{12} \\ S^{c_1}_{\mathcal{F}_{21}(\cdot,\psi_2)}\widehat{\psi_1} - c_1 m_{21} + S^{c_2}_{\mathcal{F}_{22}(\psi_1,\cdot)}\widehat{\psi_2} - c_2 m_{22} \end{pmatrix} \\ &= \begin{pmatrix} S^{c_1}_{\mathcal{F}_{11}(\cdot,\psi_2)} & S^{c_2}_{\mathcal{F}_{12}(\psi_1,\cdot)} \\ S^{c_1}_{\mathcal{F}_{21}(\cdot,\psi_2)} & S^{c_2}_{\mathcal{F}_{22}(\psi_1,\cdot)} \end{pmatrix} \begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix} - \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= S^{c}_{\mathcal{F}}(\psi)\, a - M^{c}_{\mathcal{F}}(\psi)\, c. \end{aligned}$$

Here $a = \begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix}$ is the input spectrum, $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is the vector of mean values, $S^{c}_{\mathcal{F}}(\psi)$ is the scattering operator of ψ described by an operator valued matrix 2×2, and $M^{c}_{\mathcal{F}}(\psi)$ is the 2×2 Matrix of scattering sequences for the two port. Should it be the case that $S_{\mathcal{F}_{11}}$ resp. $S_{\mathcal{F}_{21}}$ is well-defined for ψ_1 or $S_{\mathcal{F}_{12}}$ resp. $S_{\mathcal{F}_{22}}$ is well-defined for ψ_2 the corresponding formulas can be obtained by setting c_1 or c_2 to zero. This means that $c_1, c_2 = 0$ has the special meaning that ψ_1 or ψ_2 is in the domain of the unmodified scattering operator irrespective of the fact if their mean value is zero or not. This interpretation is now always used from know on.

Consequently we arrive at the following definition of the two port scattering operator.

Definition 1.6. *Let a, b, c and $\mathcal{F}$ be defined as before. Set*

$$dom\left(S_{\mathcal{F}}^{c}\right) = \{\psi \in dom\left(\mathcal{F}\right) \quad | \quad \psi_1 \in dom\left(S^{c_1}_{\mathcal{F}_{11}(\cdot,\psi_2)}\right) \cap dom\left(S^{c_1}_{\mathcal{F}_{21}(\cdot,\psi_2)}\right) \text{ and } \psi_2 \in dom\left(S^{c_2}_{\mathcal{F}_{12}(\psi_1,\cdot)}\right) \cap dom\left(S^{c_2}_{\mathcal{F}_{22}(\psi_1,\cdot)}\right)\}$$

Then the two port scattering operator is defined as the following map into equivalence classes of operators valued matrices

$$S_{\mathcal{F}}^{c} : \begin{array}{l} dom\left(S_{\mathcal{F}}\right) \to \mathbb{M}_2\left(\cup_{x \in l^2(\mathbb{Z})} \pi_x\right) \\ \psi \mapsto \begin{pmatrix} \left[S^{c_1}_{\mathcal{F}_{11}(\cdot,\psi_2)}\right]_{\sim} & \left[S^{c_2}_{\mathcal{F}_{12}(\psi_1,\cdot)}\right]_{\sim} \\ \left[S^{c_1}_{\mathcal{F}_{21}(\cdot,\psi_2)}\right]_{\sim} & \left[S^{c_2}_{\mathcal{F}_{22}(\psi_1,\cdot)}\right]_{\sim} \end{pmatrix} \end{array}$$

and the following relation between input spectrum a and output spectrum b is valid

$$b = S_{\mathcal{F}}^{c}\left(\psi\right) a - M_{\mathcal{F}}^{c}\left(\psi\right) c,$$

where $M_{\mathcal{F}}^{c}\left(\psi\right)$ denotes the 2×2 matrix of scattering sequences

$$M_{\mathcal{F}}^{c}\left(\psi\right) = \begin{pmatrix} m^{c_1}_{\mathcal{F}_{11}}\left(\psi_1\right) & m^{c_2}_{\mathcal{F}_{12}}\left(\psi_2\right) \\ m^{c_1}_{\mathcal{F}_{21}}\left(\psi_1\right) & m^{c_2}_{\mathcal{F}_{22}}\left(\psi_2\right) \end{pmatrix}.$$

As usual we omit the explicit notation of equivalence classes. The operators in the scattering operator can always be choosen as convolution operators and as such can be represented by the bi-infinite Laurent matrices. Hence two port scattering matrices are representable in block Laurent form.

1.2.1 Nonlinear Scattering in One Dimension

A system which leads to a nonlinear two port formulation is provided by the nonlinear wave equation

$$-\psi^{''} + \mathcal{V}\left(x, \psi\right) = \mathfrak{K}^2 \psi\left(x\right) \tag{4}$$

which describes the interaction of a time-harmonic scalar wave, $e^{-i\omega t}\psi\left(x\right)$, in one dimensional quantum mechanics described by the Schrödinger equation with non linear interaction. Here a prime denotes differentiation with respect to x, ψ is a possible complex valued function, $\mathfrak{K}$ denotes the wavenumber, and $\mathcal{V}\left(x, \psi\right)$ denotes the interaction.

In particular we consider the nonlinear point interaction defined by

$$\begin{aligned} \mathcal{V}\left(x, \psi\right) &= v\left(x, \psi\right) \psi, \\ v\left(x, \psi\right) \psi &= \mathfrak{f}\left(\left|\psi\left(x\right)\right|\right) \delta\left(x - c\right), \end{aligned}$$

where $\mathfrak{f} : \mathbb{R} \to \mathbb{C}$ is a continuous function and $c \in \mathbb{R}$ is the location of the nonlinear singularity. The solution of the nonlinear Schrödinger equation (NLSE) 4 is given by

$$\psi(x) = \begin{cases} a_1 e^{i\mathfrak{K}x} + b_1 e^{-i\mathfrak{K}x} & \text{for} \quad x < c, \\ b_2 e^{i\mathfrak{K}x} + a_2 e^{-i\mathfrak{K}x} & \text{for} \quad x > c, \end{cases}$$

with complex coefficients a_1, b_1, a_2, b_2 and the transfer matrix $T(a_1, b_1)$ connecting the pair a_1, b_1 to a_2, b_2 is known to be of the form [1]

$$T(a_1, b_1) = \begin{pmatrix} ge^{2ic\mathfrak{K}} & 1+g \\ 1-g & -ge^{-2ic\mathfrak{K}} \end{pmatrix}, \tag{5}$$

where $g = g(a_1, b_1) = \frac{i}{2\mathfrak{K}} \mathfrak{f}(|z|)$ and $z = a_1 e^{2ic\mathfrak{K}} + b_1$. With the help of 5 we obtain that z satisfies also the following nonlinear equation

$$z = \frac{\mathfrak{a}}{1 + \frac{i}{2\mathfrak{K}} \mathfrak{f}(|z|)}, \tag{6}$$

where $\mathfrak{a} = a_1 e^{2ic\mathfrak{K}} + a_2$, showing that $z = z(a_1, a_2)$ and after some manipulations involving Equations 5 and 6 we get

$$b_1(a_1, a_2) = z(a_1, a_2) - a_1 e^{2ic\mathfrak{K}} \tag{7}$$

$$b_2(a_1, a_2) = (z(a_1, a_2) - a_2) e^{-2ic\mathfrak{K}}. \tag{8}$$

For example, assume that f is given by a Kerr nonlinearity, $\mathfrak{f}(x) = \zeta x^2$, where we take the coupling constant $\zeta \in \mathbb{R} \setminus \{0\}$ for simplicity. Then it can be shown that Equation 6 admits the unique solution

$$z(a_1, a_2) = \mathfrak{a} \left(u - sign(\zeta)\, i \sqrt{u(1-u)} \right), \tag{9}$$

where $\mathfrak{a}$ is defined as above after Equation 6 and if $C = \frac{\zeta |\mathfrak{a}|^2}{2\mathfrak{K}} \neq 0$

$$u = \frac{1}{C} \left(\left(\frac{1}{18} \right)^{\frac{1}{3}} u_0 - \left(\frac{2}{3} \right)^{\frac{1}{3}} \frac{1}{u_0} \right), \tag{10}$$

$$u_0 = \left(9C + \sqrt{3(4 + 27C^2)} \right)^{\frac{1}{3}}.$$

In the case $C = 0$ we have to set $u = 1$.

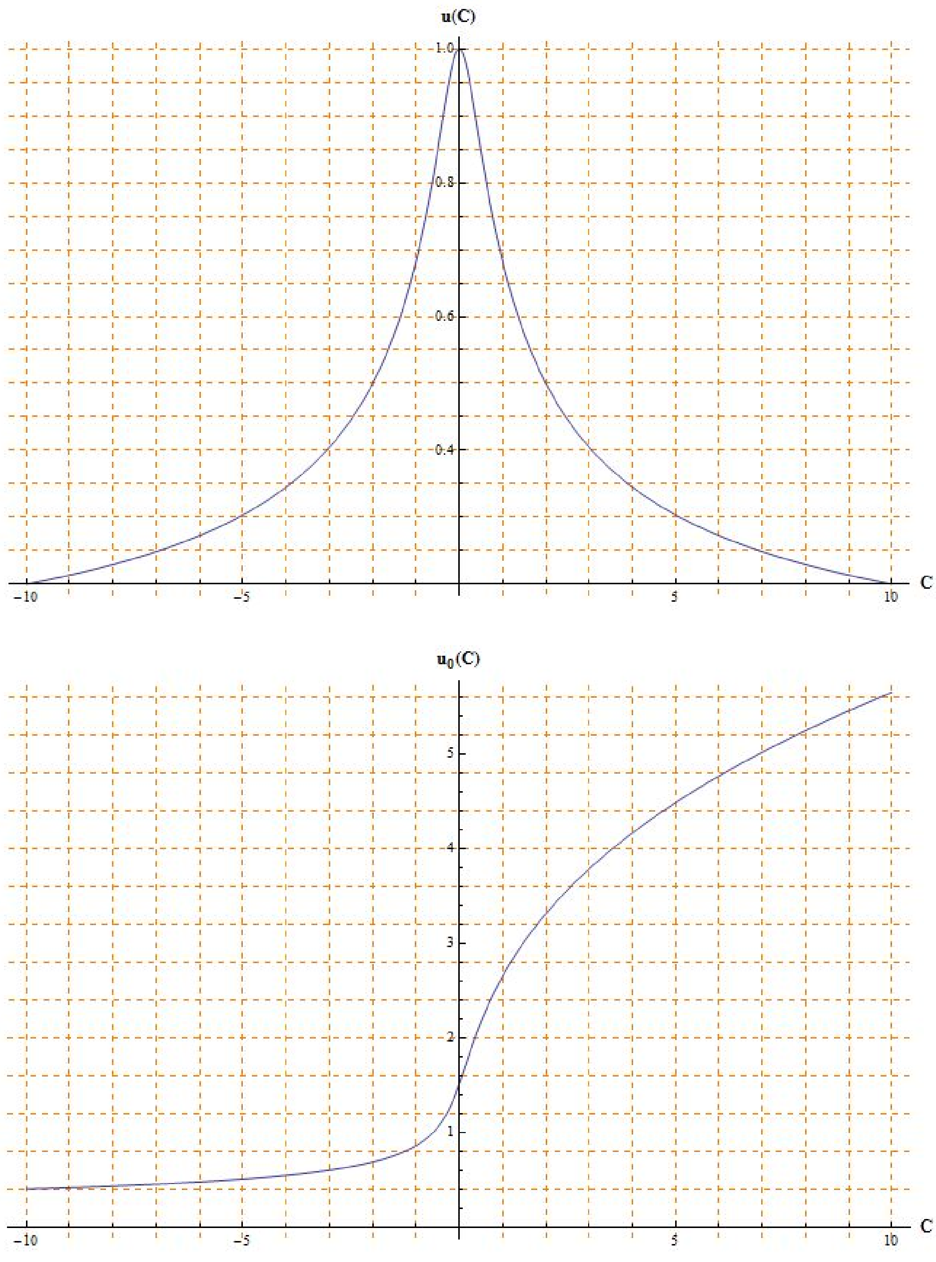

Figure 4: Functions in the Definition of $z\left(a_1, a_2\right)$

The next figures show a visualization of $z\,(a_1, a_2)$, $b_1\,(a_1, a_2)$ and $b_1\,(a_1, a_2)$ on the torus. This is achieved by setting $a_1 = \sqrt{p}Ee^{i\theta_1}$ and $a_2 = \sqrt{1-p}Ee^{i\theta_2}$ for some constants $0 < p < 1$ and $E > 0$. In the 3D visualization the smaller radius of the torus is given by $\sqrt{\min\,(p, 1-p)}E$ and the greater one by $\sqrt{\max\,(p, 1-p)}E$. Here complex numbers are coded by the following color scheme. The hue represents the phase of complex numbers and the brightness of the corresponding color encodes their absolute value. Black corresponds to zero. The free parameters were set to $\mathfrak{K} = 2$, $c = 5$, and $\zeta = 3$.

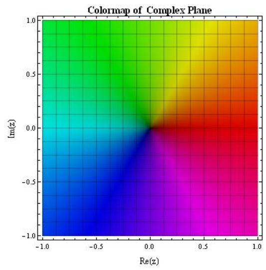

Figure 5: Color Code of Complex Numbers

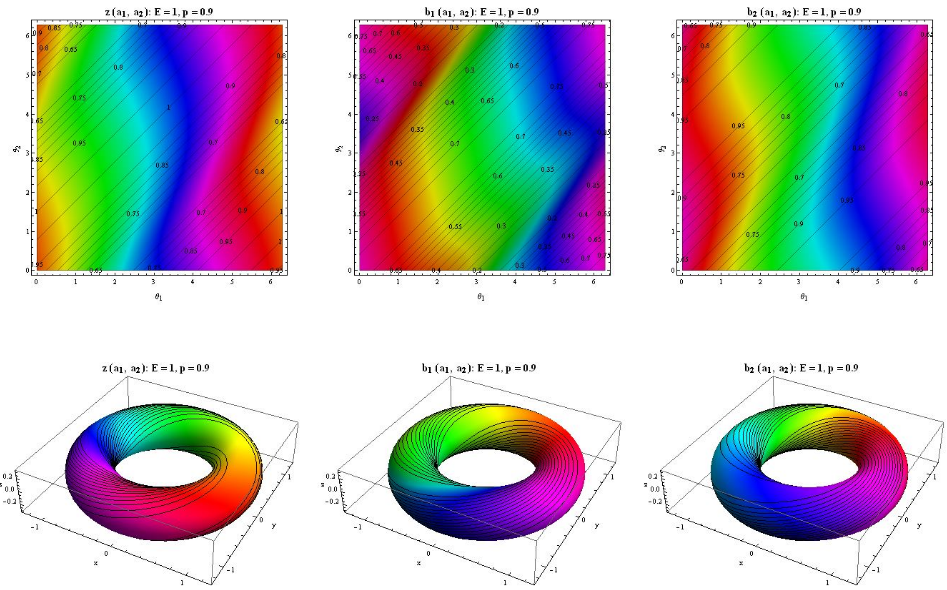

Figure 6: Complex Scattering Functions for NLSE, $\mathfrak{K} = 2$, $c = 5$, and $\zeta = 3$

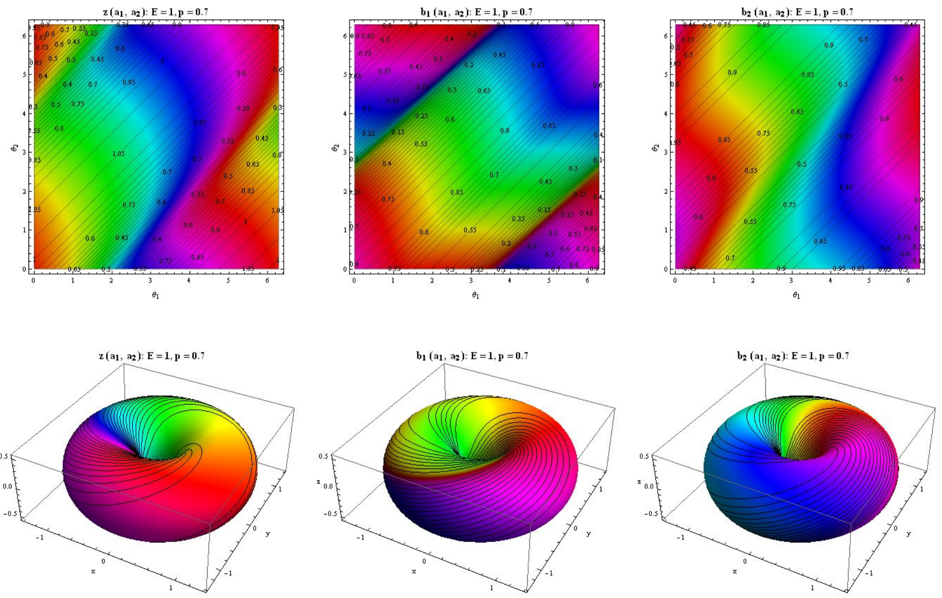

Figure 7: Complex Scattering Functions for NLSE, $\mathfrak{K} = 2$, $c = 5$, and $\zeta = 3$

1.2.2 The Scattering Matrix for the NLSE

I principle a nonlinear scattering matrix for Equations 7, 8 could be derived by brute force comparism. However this would not take into account the fact that due to the defining relation $b\,(a) = S\,(a)\,a$, $a \in \mathbb{C}^2$ not all information contained in the matrix S is needed to obtain b_1, b_2 since these are already obtained by evaluation at the parameter a of S. Moreover any systematic method for construction a scattering matrix from the knowledge of $b\,(a_1, a_2)$ should give us the usual scattering matrix in the linear case. This is made more precise in the following lemma.

Lemma 1.7. *Suppose that Ω is a domain in $\mathbb{C}^2$ containing 0 and $b : \Omega \setminus \{0\} \to \mathbb{C}^2$ a vector valued function. Let be $a \in \Omega$ with $a_1, a_2 \neq 0$.*

Then there are matrices S_0, D and N depending on a, s. t. D is diagonal, $a \in \ker(N)$, and $S = S_0 + D + N$. Consequently the redundant part is given by N. S_0 and D are explicitely obtained by

$$S_0 = \begin{pmatrix} \frac{b(a_1,0)}{a_1} & \frac{b(0,a_2)}{a_2} \end{pmatrix}, \tag{11}$$

$$D = diag(a)^{-1} diag\,(\Delta b) \quad \textit{where} \tag{12}$$

$$\Delta b = b\,(a_1, a_2) - b_1\,(a_1, 0) - b_2\,(0, a_2)\,, \tag{13}$$

i.e. these matrices are determined by the output function b. Morover $S_0 = S$, $\Delta b = 0$ and $N = 0$ if $b = Sa$ for some constant matrix $S \in \mathbb{M}_2\,(\mathbb{C})$.

Thus using Equations 11 and 12 together with Equation 13 we obtain

$$\begin{aligned}\mathfrak{S}(a) &:= S_0 + D \\ &= \begin{pmatrix} \frac{z(a_1,a_2)-z(0,a_2)}{a_1} - e^{2ic\mathfrak{K}} & \frac{z(0,a_2)}{a_2} \\ \frac{z(a_1,0)}{a_1} e^{-2ic\mathfrak{K}} & \left(\frac{z(a_1,a_2)-z(a_1,0)}{a_2} - 1\right) e^{-2ic\mathfrak{K}} \end{pmatrix}. \quad (14)\end{aligned}$$

A quick check shows that this matrix applied to a produces the expressions on the right side given in Equation 7 and 8 and can be used as scattering matrix in canonical form, i.e. which is determined by the three output measurements $b(a_1, 0)$, $b(0, a_2)$, and $b(a_1, a_2)$.

Remark 2. Observe that the term redundant has only a precise meaning with respect to the properties we require for the scattering matrix, e.g. if we require as in the last lemma that it should be determined only by the output measurements we can omit N. However if we require further properties as for instance continiuity for zero inputs we can take advantage of the additional degree of freedom to add some appropriate matrix N to obtain the required property.This is done in the following subsection.

1.2.3 Extension of the NLSE Scattering Matrix to Zero Inputs

As long as we are concerned with Equation 7 an 8 which make also sense on $(\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$ the matrix function $a \mapsto \mathfrak{S}(a)$ could be extended in a trivial but rather arbitrary way to be compatible with these equations. However this can be also achieved in a unique way by continuity.

To be more precise we are looking for a continuous representative in the equivalence class of $\mathfrak{S}(a)$. Such an extension is important, because it defines also the zero-th order term in a perturbation expansion of the scattering matrix.

For that purpose we define

$$f: \begin{array}{l} \mathbb{R} \to \mathbb{D} \\ t \mapsto u(t) - sign(t)\, i\sqrt{u(t)(1-u(t))} \end{array},$$

where the function u is defined as in Equation 10 and $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Observe that f, f', and $\frac{1}{2}f''$ map into the unit disk $\mathbb{D}$ as can be recognized from the following figures.

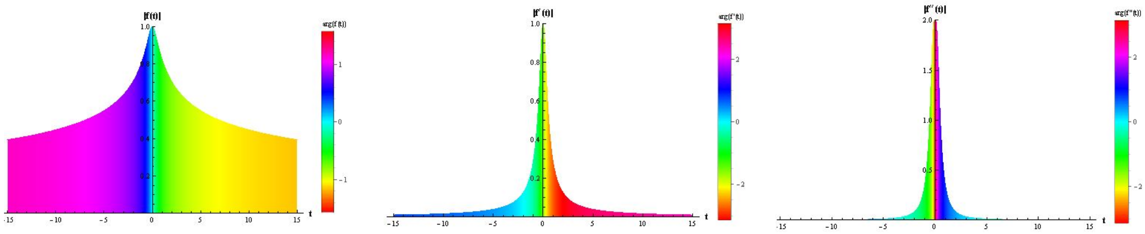

Figure 8: Absolute Value and Argument of the Function f, f', and f''

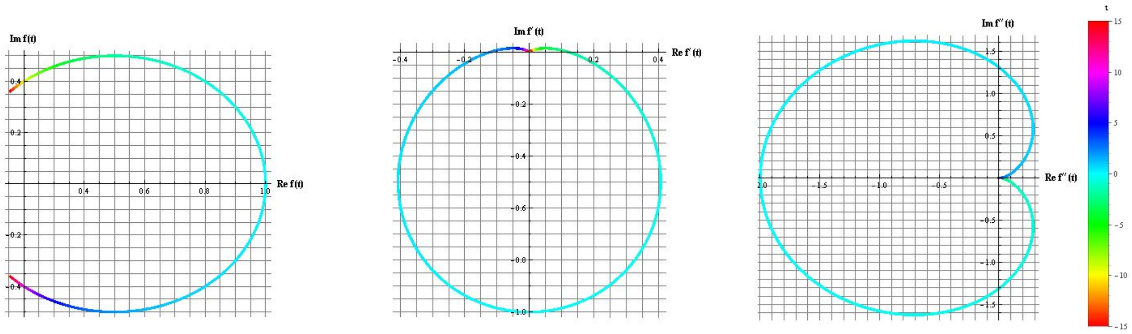

Figure 9: Phase Space Plot of the Function f, f', and f''

Next recall that $C : \mathbb{C}^2 \to \mathbb{R}$ is given by $C(a_1, a_2) = \frac{\zeta \left| a_1 e^{2ic\mathfrak{K}} + a_2 \right|^2}{2\mathfrak{K}}$. Then with the function $h : \mathbb{C}^2 \to \mathbb{D}$, $h = f \circ C$, we can write by the definition of z in Equation 9 and since $sign(C(a_1, a_2)) = sign(\zeta)$

$$\begin{aligned} z(a_1, a_2) &= \left(a_1 e^{2ic\mathfrak{K}} + a_2 \right) h(a_1, a_2) \quad \text{implying} \\ \frac{z(a_1, 0)}{a_1} &= e^{2ic\mathfrak{K}} h(a_1, 0) \quad \text{and} \\ \frac{z(0, a_2)}{a_2} &= h(0, a_2). \end{aligned}$$

Since $\lim_{a_1 \to 0} h(a_1, 0) = \lim_{a_2 \to 0} h(0, a_2) = 1$ it follows that $\lim_{a_1 \to 0} \frac{z(a_1,0)}{a_1} = e^{2ic\mathfrak{K}}$ and $\lim_{a_2 \to 0} \frac{z(0,a_2)}{a_2} = 1$, in particular this means that z is a partially holomorphic function at the origin of $\mathbb{C}^2$. Unfortunately this cannot be extended to values $a_1, a_2 \neq 0$ because the limits $\lim_{a_1 \to 0} \frac{\Delta z(0,a_2)}{a_1} = \lim_{a_1 \to 0} \frac{z(a_1,a_2) - z(0,a_2)}{a_1}$ and $\lim_{a_2 \to 0} \frac{\Delta z(a_2,0)}{a_2} = \lim_{a_2 \to 0} \frac{z(a_1,a_2) - z(a_1,0)}{a_2}$ do not exist. This is already indicated in the following plot for the first difference quotient where the dependence on the phase of a_1 for small a_1 becomes clearly visible and is actually caused by the nonanalytic form of the Kerr nonlinearity. To identitfy the terms which cause this

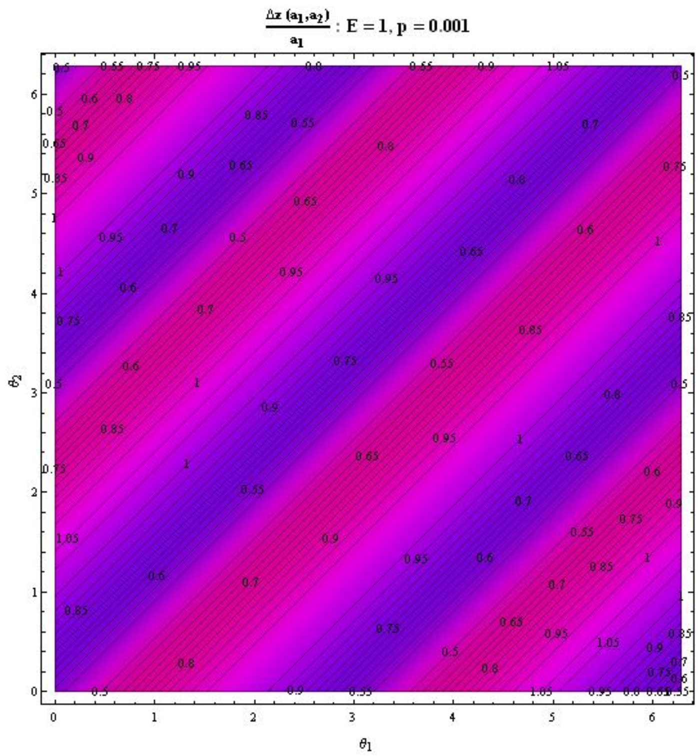

Figure 10: First Difference Quotient for small a_1, $\mathfrak{K} = 2$, $c = 5$, and $\zeta = 3$

phase dependence in the difference quotients one has to apply a Taylor series expansion in the sense of the Wirtinger calculus for $h(a_1, a_2)$ at $a_1 = 0$ resp. $a_2 = 0$ up to a second order remainder term. This shows that we have to add the following matrix

$$N(a) = \begin{pmatrix} -\frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(0, a_2)) e^{-2ic\mathfrak{K}} a_2^2 \frac{a_1^*}{a_1} & \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(0, a_2)) e^{-2ic\mathfrak{K}} a_1^* a_2 \\ \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(a_1, 0)) e^{2ic\mathfrak{K}} a_1 a_2^* & \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(a_1, 0)) e^{2ic\mathfrak{K}} a_1^2 \frac{a_2^*}{a_2} \end{pmatrix}$$

to the scattering matrix $\mathfrak{S}(a)$ to obtain a regularized scattering matrix with components

$$\begin{aligned}
\mathfrak{S}_{11}^r(a) &= (h(a_1, a_2) - 1) e^{2ic\mathfrak{K}} + a_2 \frac{\Delta h(0, a_2) - \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(0, a_2)) e^{-2ic\mathfrak{K}} a_1^* a_2}{a_1}, \\
\mathfrak{S}_{12}^r(a) &= h(0, a_2) + \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(0, a_2)) e^{-2ic\mathfrak{K}} a_1^* a_2, \\
\mathfrak{S}_{21}^r(a) &= h(a_1, 0) + \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(a_1, 0)) e^{2ic\mathfrak{K}} a_1 a_2^*, \\
\mathfrak{S}_{22}^r(a) &= (h(a_1, a_2) - 1) e^{-2ic\mathfrak{K}} + a_1 \frac{\Delta h(a_1, 0) - \frac{\zeta}{2\mathfrak{K}} \frac{\partial f}{\partial t} (C(a_1, 0)) e^{2ic\mathfrak{K}} a_1 a_2^*}{a_2}
\end{aligned}$$

for all $a_1, a_2 > 0$, which can now continuously extended to $(\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$ by setting

$$\begin{aligned}
&\mathfrak{S}^r(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\mathfrak{S}^r(a_1, 0) = \\
&\quad \begin{pmatrix} (h(a_1, 0) - 1) e^{2ic\mathfrak{K}} & 1 \\ h(a_1, 0) & \left(h(a_1, 0) + C(a_1, 0) \frac{\partial f}{\partial t} (C(a_1, 0)) - 1 \right) e^{-2ic\mathfrak{K}} \end{pmatrix}, \\
&\mathfrak{S}^r(0, a_2) = \\
&\quad \begin{pmatrix} \left(h(0, a_2) + C(0, a_2) \frac{\partial f}{\partial t} (C(0, a_2)) - 1 \right) e^{2ic\mathfrak{K}} & h(0, a_2) \\ 1 & (h(0, a_2) - 1) e^{-2ic\mathfrak{K}} \end{pmatrix}.
\end{aligned}$$

Remark 3. Obviously by construction $\mathfrak{S}^r(a) a = b(a)$ for all $a \in \mathbb{C}^2$. Moreover it is easy to check that the X parameters are connected to the linearization of the matrix $\mathfrak{S}^r(a)$, e.g. if a_2 is considered as the large signal $\mathfrak{S}_{11}^r(a)$ defines the first X^S parameter for b_1 and $\mathfrak{S}_{12}^r(a)$ defines after multiplication with a_2 the X^F parameter and the factor of a_1^* defines the X^T parameter for b_1, $\mathfrak{S}_{21}^r(a)$ contains no information about any X parameters and is simply needed to cancel out terms, and finally $\mathfrak{S}_{22}^r(a)$ contains all information about X parameters for b_2.

Remark 4. For the NLSE we can define the two port operator $\mathcal{F}$ by $dom(\mathcal{F}) = L^2\left(-\frac{\pi}{\mathfrak{K}}, \frac{\pi}{\mathfrak{K}}\right)^2$ with $\mathcal{F}(\psi) = \left(\mathfrak{S}_{ij}^r \left(\langle \widehat{\psi_1}, \delta_1 \rangle, \langle \widehat{\psi_2}, \delta_{-1} \rangle \right) \psi_j \right)_{i,j=1,2}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $l^2(\mathbb{Z})$.Then the scattering operator for $\mathcal{F}$ is given by

$dom\left(\mathcal{S}_{\mathcal{F}}\right) = L^2\left(-\frac{\pi}{\mathfrak{K}}, \frac{\pi}{\mathfrak{K}}\right)^2$ and $\mathcal{S}_{\mathcal{F}}\left(\psi\right) = \left(\mathfrak{S}^r_{ij}\left(\langle\widehat{\psi_1}, \delta_1\rangle, \langle\widehat{\psi_2}, \delta_{-1}\rangle\right)\delta_0 *\right)_{i,j=1,2}$ as expected.

1.3 Conclusion

We suggested an extension of the nonlinear large signal scattering S parameters which were defined in frequency domain in a finite dimensional setting ([1] [2], [3], [4],). On the contrary we tried to start in time domain and supposed that there is some nonlinear operator which gives the relation of input (periodic stimulus) and output states (periodic steady state) of the system. In the most general case this could be thought as the solution operator to some ordinary nonlinear differential (ODE) or differential algebraic equation (DAE). How the scattering operator relates to the particular form of the ODE or DAE was not investigated in this article and has to be investigated in the future.

Now in this setting it possible to relate a nonlinear scattering operator to the periodic steady state and to the perturbed periodic steady state obtained by a perturbation of the periodic stimulus regardless if the perturbation is small or not. As was demonstrated in the case of the nonlinear Schrödinger equation with Kerr nonlinearity the scattering operator can be even defined for arbitrary input states. Consequently this means that changes in the output spectrum are intimately connected with changes in the nonlinear scattering operator which demands for a perturbation theory of the nonlinear scattering operator, which seems to be not well developed in the literature.

Nonlinear scattering operators are determined by a scattering sequence or in the multiport case by a set of scattering sequences and given as convolution operators or block convolution operators meaning that much less data would be needed to determine the nonlinear scattering operator. This is somehow unspected from the work given in [2], [3], [4], but should be connected to the nonuniqueness of the nonlinear scattering operator and the appearance of the operator N. Therefore it is important to know if the solutions of ODEs, resp. DAEs are in the domain of the scattering operator defined in this article.

For the nonlinear Schrödinger equation with Kerr nonlinearity we showed how the nonlinear scattering matrix in frequency domain could be derived from a given set of nonlinear equations in frequency domain. Moreover this matrix was continuously extended such that a linearization of the scattering matrix can be carried out. It is currently an open questions, how this procedure could be done for more ports or frequencies.

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